Loday constructions for Tambara functors

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joint work with Ayelet Lindenstrauss and Foling Zou

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What are they?

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  we have $M^*(\delta) \circ M_*(\gamma) = M_*(\beta) \circ M^*(\alpha)$,
- for every pair of finite G-sets X and Y, applying M_* to $X \to X \sqcup Y \leftarrow Y$ gives the component maps of an isomorphism $\underline{M}(X) \oplus \underline{M}(Y) \cong \underline{M}(X \sqcup Y)$.

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$\underline{A}$ is initial in Tamb_G and a unit for the so-called box product of G-Mackey functors, $\square$.

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Theorem [Kristen Mazur 2013, Rolf Hoyer 2014] There is a functor

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2. There is a natural isomorphism $X \otimes (Y \otimes \underline{R}) \cong (X \times Y) \otimes \underline{R}$.
3. On the category with objects finite sets with trivial G-action and morphisms consisting only of isomorphisms, the functor restricts to exponentiation $X \otimes \underline{R} = \Box_{x \in X} \underline{R}$.

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The pair (N_H^G, i_H^*) is an adjoint functor pair.

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The proof is by direct inspection, where we use the fact that $\underline{R}^c \Box \underline{R}^c \cong \underline{(R \otimes R)}^c$.

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$$\mathcal{L}_X^{C_2}(\underline{\mathbb{Z}}^c) \cong \begin{cases} \underline{\mathbb{Z}}^c, & \text{if } X^{C_2} \neq \varnothing, \\ \underline{A}, & \text{if } X^{C_2} = \varnothing. \end{cases}$$

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We saw that $\underline{A}$ is the initial object in Tamb_{C_2} and the ring of integers is initial in the category of commutative rings. Therefore

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If there is a fixed point somewhere, then we have one in every simplicial level. A fixed point corresponds to the orbit C_2/C_2, hence there we get $\underline{\mathbb{Z}}^c$. The claim follows from $\underline{\mathbb{Z}}^c \Box \underline{A} \cong \underline{\mathbb{Z}}^c$. $\square$

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& \gamma^2 x_0 \arrow[dl, dotted, bend right] & \\
\gamma^{-1} x_0 \arrow[dr, bend right, "e_0"'] & & \gamma x_0 \arrow[ul, bend right, "\gamma^2 e_0"'] \\
& x_0 \arrow[ur, bend right, "\gamma e_0"'] &
\end{tikzcd}
$$

We have $(S^1_{\mathrm{rot}})_k = \{C_n \cdot x_k^0, C_n \cdot x_k^1, \cdots, C_n \cdot x_k^k\}$, where

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The simplicial identities imply that

$$\begin{aligned} d_j(x_k^0) =& x_{k-1}^0, \\ d_j(x_k^i) =& \begin{cases} x_{k-1}^{i-1} & 0 \leq j \leq i-1 \\ x_{k-1}^i & i \leq j \leq k \text{ and } i \neq k \end{cases} \\ d_k(x_k^k) =& \gamma^{-1} x_{k-1}^0. \end{aligned}$$

So for a C_n-Tambara functor $\underline{R}$ with $R := i_e^* \underline{R}$, there is

$$\mathcal{L}^{C_n}_{S^1_{\mathrm{rot}}}(\underline{R})_k = \square_{0 \leq i \leq k}(C_n \otimes \underline{R}) = (N_e^{C_n} R)^{\square(k+1)},$$

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and $d_i \colon (N_e^{C_n} R)^{\square(k+1)} \to (N_e^{C_n} R)^{\square k}$ is

$$\begin{aligned} d_i &= \mathrm{id}^i \square \mu \square \mathrm{id}^{k-i} && \text{for } 0 \leq i < k \\ d_k &= (\mu \square \mathrm{id}^{k-1}) \circ (\gamma^{-1} \square \mathrm{id}^k) \circ \tau \end{aligned}$$

where $\mu \colon (N_e^{C_n} R)^{\square 2} \to N_e^{C_n} R$ is the multiplication and $\tau \colon (N_e^{C_n} R)^{\square(k+1)} \to (N_e^{C_n} R)^{\square(k+1)}$ moves the last coordinate to the front.

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In particular, for $K = C_n$:

$$\mathcal{L}^{C_n}_{S^1_{\mathrm{rot}}/C_n}(\underline{R}) \cong \underline{\mathrm{HC}}_{C_n}^{C_n}(\underline{R}) = \underline{\mathrm{HC}}^{C_n}(\underline{R}).$$